

## Uniform Linear Acceleration

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# Uniform Linear Acceleration

Uniform linear acceleration is the simplest gedanken experiment in general relativity.

Objects in uniform linear acceleration, at fixed height above and seen from below experience faster time passage. Objects in uniform linear acceleration, at fixed height below and seen from above experience slower time passage. 1930s radio operators using the new crystal radiotelephones between mountains and valleys, called this time distortion *Einstein Effect*.

## Accelerated Point Object Worldline

Consider an object, initially at rest in an inertial reference frame,  $irf_0$ , that undergoes constant acceleration,  $A$ , for an infinitesimal interval,  $dt_0$ , after which it is at rest in an inertial reference frame,  $irf_1$ , and a third inertial reference frame,  $irf_r$ , with speed,  $v$ , relative to  $irf_0$ .

Represent events as (time, distance, speed) .

$irf_r$	$irf_0$	$irf_1$	Notes
$(0, , v)$	$(0, , 0)$	$(0, , -Adt_1)$	initial speed of object
$(dt_r, , \frac{v + Adt_0}{1 + \frac{vAdt_0}{c^2}})$	$(dt_0, , Adt_0)$	$(dt_1, , 0)$	final speed of object
$(dt_r, , v + dt_0 A \frac{1 - \frac{v^2}{c^2}}{1 - (\frac{vAdt_0}{c^2})^2})$			

Determine observed speed,  $v$ , in  $irf_r$ , after constant acceleration in rest frame,  $irf_0$ , at time,  $t_0$ , in  $irf_0$  . To do so, integrate an infinite chain of inertial reference frames in infinitesimal motion, and integrate.

$\frac{dv}{dt_0} = A \left( \frac{\sqrt{c^2 - v^2}}{c} \right)^2$	acceleration observed in rest frame
$\frac{dv}{dt_r} = A \left( \frac{\sqrt{c^2 - v^2}}{c} \right)^3$	

$v = c \tanh\left(\frac{At_0}{c}\right)$	speed observed in rest frame
$v = c \frac{\frac{At_r}{c}}{\sqrt{1 + \left(\frac{At_r}{c}\right)^2}}$	
$\frac{dt_r}{dt_0} = \frac{c}{\sqrt{c^2 - v^2}}$	lorentz factor between rest and accelerated frame
$\frac{dt_r}{dt_0} = \cosh\left(\frac{At_0}{c}\right)$	
$t_r = \frac{c}{A} \sinh\left(\frac{At_0}{c}\right)$	accelerated frame time vs rest frame time
$t_0 = \frac{c}{A} \ln\left(\frac{At_r}{c} + \sqrt{\left(\frac{At_r}{c}\right)^2 + 1}\right)$	
$x_r = \frac{c^2}{A} \left(\cosh\left(\frac{At_0}{c}\right) - 1\right)$	rest frame position
$x_r = \frac{c^2}{A} \left(\sqrt{1 + \left(\frac{At_r}{c}\right)^2} - 1\right)$	

## Summary

Point object uniform linear acceleration worldline follows.

(time, position, inertial speed)

accelerated frame:  $(t_0, 0, 0)$

accelerated frame:  $\left(\frac{c}{A} \ln\left(\frac{At_r}{c} + \sqrt{\left(\frac{At_r}{c}\right)^2 + 1}\right), 0, 0\right)$

inertial frame:  $\left(\frac{c}{A} \sinh\left(\frac{At_0}{c}\right), \frac{c^2}{A} \left(\cosh\left(\frac{At_0}{c}\right) - 1\right), c \tanh\left(\frac{At_0}{c}\right)\right)$

inertial frame:  $\left(t_r, \frac{c^2}{A} \left(\sqrt{1 + \left(\frac{At_r}{c}\right)^2} - 1\right), c \frac{\frac{At_r}{c}}{\sqrt{1 + \left(\frac{At_r}{c}\right)^2}}\right)$

## Gravitational Red Shift

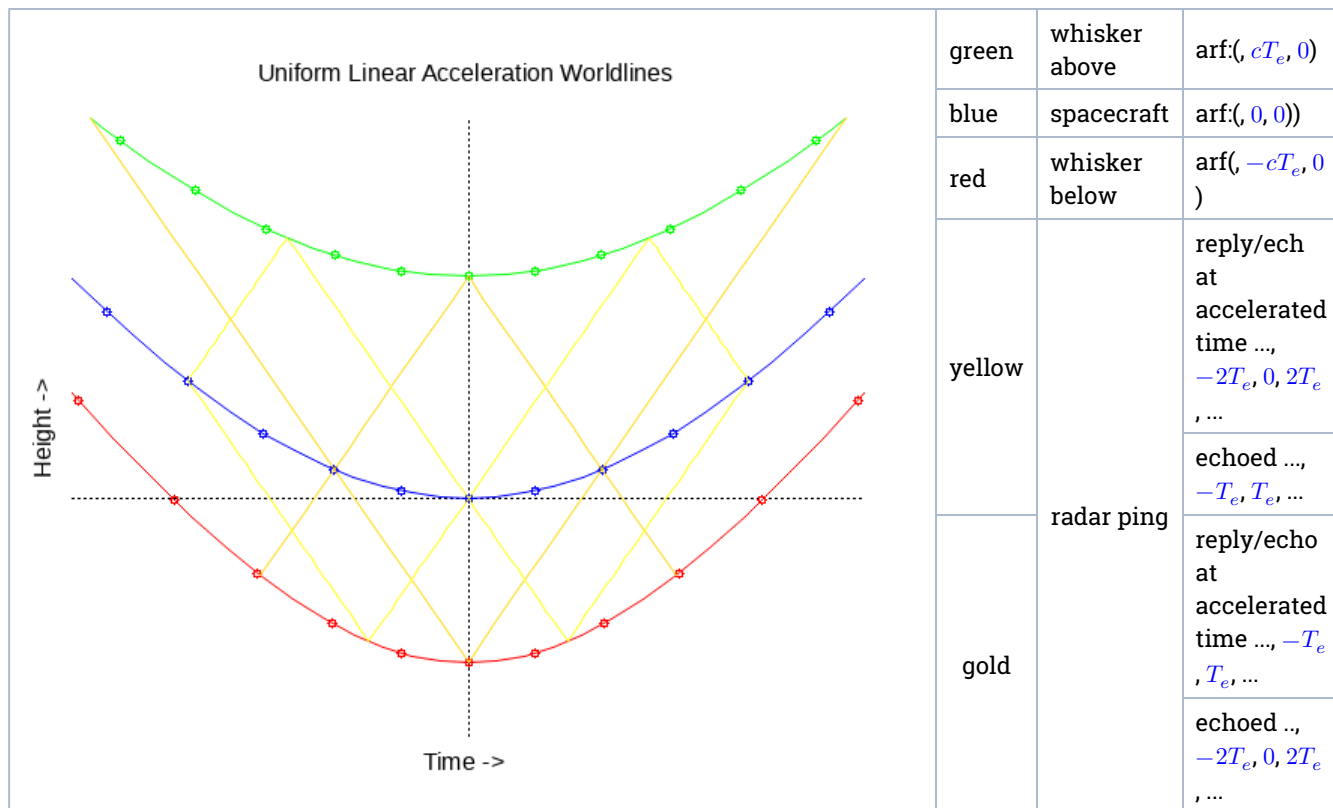
Gravitational Red Shift, known to radio operators since the 1930s as Einstein Effect, relates two accelerated reference frames.

### Accelerated Fixed Separation Worldline

Consider the worldline of three point objects in linear uniform acceleration, e.g. two whiskers holding fixed station equidistant above and below a spacecraft. (Whiskers are unmanned ancillary vehicles that autonomously track a larger manned vehicle, submarine, automobile, aircraft, spacecraft, for sensing, navigation, diagnostics, ... We can imagine the whiskers are targets for the spacecraft radar and a pair of radar pings (photons), bounce back and forth between them.

The spacecraft (blue) worldline is point object uniform linear acceleration. At fixed distance, therefore time-invariant, the whisker (red and green) worldlines are also point object uniform linear acceleration.

Two events including time, height, and speed, are sufficient to fit a point object uniform linear acceleration worldline. Two convenient events are (gold) echo at rest and echo after (yellow) ping at rest.



Consider ping echoed at rest.

$t_e = \frac{c}{A_0} \sinh\left(\frac{A_0 T_e}{c}\right)$	ping/reply inertial time and height
$h_e = \frac{c^2}{A_0} \left( \cosh\left(\frac{A_0 T_e}{c}\right) - 1 \right)$	
$h_{0b} = h_e - ct_e$	echo inertial height at rest
$= \frac{c^2}{A_0} \left( \cosh\left(\frac{A_0 T_e}{c}\right) - \sinh\left(\frac{A_0 T_e}{c}\right) - 1 \right)$	
$= \frac{c^2}{A_0} \left( e^{-\frac{A_0 T_e}{c}} - 1 \right)$	
$h_{0a} = \frac{c^2}{A_0} \left( e^{\frac{A_0 T_e}{c}} - 1 \right)$	

Consider ping and reply at rest.

$c \tanh\left(\frac{A_b T_b}{c}\right) = c \tanh\left(\frac{A_0 T_e}{c}\right) = c \tanh\left(\frac{A_a T_a}{c}\right)$	fixed separation implies same speed
$\frac{A_b T_b}{c} = \frac{A_0 T_e}{c} = \frac{A_a T_a}{c}$	monotonic function implies same argument
$h_{0b} + \frac{c^2}{A_b} \left( \cosh\left(\frac{A_b T_b}{c}\right) - 1 \right) = -c \frac{c}{A_b} \sinh\left(\frac{A_b T_b}{c}\right)$	ping from rest to whisker fixed below
$0 = \frac{c^2}{A_0} \left( e^{-\frac{A_0 T_e}{c}} - 1 \right) + \frac{c^2}{A_b} \left( e^{\frac{A_b T_b}{c}} - 1 \right)$	
$0 = \frac{c^2}{A_0} e^{-\frac{A_0 T_e}{c}} \left( 1 - e^{\frac{A_0 T_e}{c}} \right) - \frac{c^2}{A_b} \left( 1 - e^{\frac{A_b T_b}{c}} \right)$	
$0 = \frac{c^2}{A_0} e^{-\frac{A_0 T_e}{c}} - \frac{c^2}{A_b}$	Divide by same function of same argument

$A_b = A_0 e^{-\frac{A_0 c T_e}{c^2}}$	
$A_a = A_0 e^{-\frac{A_0 c T_e}{c^2}}$	

## Summary

Summarizing, a line of point objects at equal accelerated height intervals experiencing uniform linear acceleration will observe

inertial rest height distributed below (green) at exponentially decreasing intervals,

approaching an event horizon at  $-\frac{c^2}{A_0}$

inertial rest height distributed above (red) at exponentially increasing intervals

inertial time scaled exponentially by accelerated height, slower (fewer ticks) below and faster (more ticks) above

inertial time scaled faster (more ticks) by hyperbolic cosine relative to accelerated time since rest.

## Acceleration Event Horizon

As fixed separation increases as seen below from accelerating point, inertial height approaches a light cone with vertex  $\frac{c^2}{A_0}$  below rest event.

Events below this light cone are invisible, below negative infinity accelerated height, as seen by accelerating point. This manifold is called an **Event Horizon**.

All unaccelerated points, seen from an accelerated point,

1. appear below
  - near infinite depth
  - initially frozen, unchanging
  - climbing near lightspeed
2. climb
  - quicken, begin to evolve slowly at increasing rate
  - slowing climb rate
3. peak
  - normal rate of change
  - momentarily rest
4. fall
  - evolve slowly at decreasing rate
  - increasing fall rate
5. disappear below
  - near infinite depth
  - finally frozen, unchanging
  - falling near lightspeed

## Accelerated Reference Frame

An accelerated reference frame is characterized by **frame acceleration**,  $A_0$ , at its origin. At all other events in accelerated frame,

acceleration is in the same direction as frame acceleration

acceleration magnitude decreases exponentially  $A(h) = e^{-\frac{A_0 h}{c^2}}$  along radar height component parallel to frame acceleration

time rate relative to origin increases exponentially  $\frac{dt}{dt_0} = e^{\frac{A_0 h}{c^2}}$  along radar height component parallel to

frame acceleration

An accelerated reference frame sees a subset of an inertial reference frame. The accelerated reference frame space, and origin, map respectively to the inside of inertial reference frame **Event Horizon** cone and event. The event horizon cone axis passes through the accelerated frame origin in the direction of frame acceleration, and the cone tip is  $\frac{c^2}{A_0}$  below the origin.

## Formulae

### Special Relativity (collinear speed addition)

$irf_r$	$irf_0$	$irf_1$	Notes
$c$	$c$	$c$	Lightspeed is identical in all inertial reference frames
0	$v_a$	$\frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}$	speed of object observed in different frames
$-v_a$	0	$v_b$	
$-\frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}$	$-v_b$	0	
$x_r$	$x_0$	$x_1$	object position observed in different frames
$dt_r$	$dt_0$	$dt_1$	interval between events

$\frac{dt_r}{dt_0} = \frac{c}{\sqrt{c^2 - v_a^2}}$	lorentz factor
$\frac{dt_r}{dt_1} = \frac{c}{\sqrt{c^2 - v_b^2}}$	
$\frac{dt_r}{dt_1} = \frac{c}{\sqrt{c^2 - \left(\frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}\right)^2}}$	

### Hyperbolic Trigonometric Functions

$\sinh(x) = \frac{e^x - e^{-x}}{2}$	hyperbolic sine
$\cosh(x) = \frac{e^x + e^{-x}}{2}$	hyperbolic cosine
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$	hyperbolic tangent

### Inverse Hyperbolic Trigonometric Functions

$a \sinh(x) = \ln(x + \sqrt{x^2 + 1})$	hyperbolic antisine
$a \cosh(x) = -\ln(x - \sqrt{x^2 - 1})$	hyperbolic anticostine
$a \tanh(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$	hyperbolic antitangent

### Hyperbolic Trigonometric Identities

$1 = \sec^2(x) + \tanh^2(x)$	hyperbolic secant rule
$1 = \cosh^2(x) - \sinh^2(x)$	hyperbolic sine rule
$e^x = \cosh(x) + \sinh(x)$	
$e^{-x} = \cosh(x) - \sinh(x)$	

## Hyperbolic Trigonometric Function Derivatives

$\frac{d}{dx} \sinh(x) = \cosh(x)$
$\frac{d}{dx} \cosh(x) = \sinh(x)$
$\frac{d}{dx} \tanh(x) = \frac{1}{(\cosh(x))^2}$

## Software

[uniform\\_linear\\_acceleration.wmx algebra checker](#)

[uniform\\_linear\\_acceleration.bash diagram generator](#)

Reference/Uniform Linear Acceleration (last edited 2022-02-28 05:06:42 by wiki)